ASYMPTOTIC INTEGRATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH A SMALL PRINCIPAL PART

(ASIMPTOTICHESKOE INTEGRIROVANIE LINEINYKH DIFFERENTSIAL'NYKH URAVNENII V CHASTNYKH PROIZVODNYKH S MALOI GLAVNOI CHAST'IU)

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This article deals with the same type of problems of the theory of linear differential equations in two independent variables as the earlier work [1]: the boundary value problem for the elliptic equation, Cauchy's problem for the equation of the hyperbolic type, and the problem on the construction of a particular integral of an equation of arbitrary type. As in [1], we assume that the boundary values of the solution function and its derivative (or the free term of the equation) depend on a large parameter k, and represent rapidly oscillating functions. In contrast with [1], we here assume that a small parameter h appears in the coefficient of the derivatives of highest order. We investigate the nature of the solutions of the problems mentioned, and give various methods for constructing approximations to these solutions. These methods vary according to the relationships existing between parameters k and h. We assume the reader to be familiar with the content of the earlier work [1].

The method on which the present work is based was used in the author's monograph [2] on the theory of elastic shells. Certain of the results presented here have already been formulated in the said monograph, but were there stated as preliminary announcements. Here, relegating the question of the efficiency of the method to second place, the author attempts as far as possible to define the conditions in which the method is valid.

1. 1. Let us consider the equation

$$hN(\Phi) + L(\Phi) = 0 \tag{1.1}$$

where h is a small constant parameter, while L and N are differential operators of order l and n respectively (it will always be assumed that

$n > l$):

$$L = \sum_{\nu=0}^{\nu=l} \sum_{j=0}^{j=\nu} a_{j,\nu-j}^{(\nu)} \frac{\partial^\nu}{\partial\alpha^j \partial\beta^{\nu-j}}, \qquad N = \sum_{\nu=0}^{\nu=n} \sum_{j=0}^{j=\nu} b_{j,\nu-j}^{(\nu)} \frac{\partial^\nu}{\partial\alpha^j \partial\beta^{\nu-j}}$$

For the time being we assume only that the coefficients of both operators are sufficiently smooth. Additional hypotheses will be imposed on L and N later.

The equation (1.1) will be solved under boundary conditions depending on a large parameter k. We establish the connection between k and h by the formula

$$k = h^{-t} \tag{1.2}$$

where t is a positive member.

2. We will seek integrals of the equation (1.1) of the form

$$\Phi = \Phi_* e^{kf} \qquad \left(f = f_0 + \sum_{\lambda=0}^{\lambda=\zeta-\varkappa-1} k^{-\frac{\varkappa+\lambda}{\zeta}} f_{\varkappa+\lambda},\ \Phi_* = \sum_{u=0}^{u=R}{}^{*} k^{-u}\Phi_u\right) \tag{1.3}$$

Here χ and ζ are positive numbers such that $\chi < \zeta$; the symbol Σ^* (here and subsequently designates summation over all values of the appropriate index having the form $\sigma + r/\zeta$ (σ, r are nonnegative integers),

$$f_0,\ f_\varkappa,\ f_{\varkappa+1}, \ldots, f_{\zeta-1},\ \Phi_0,\ \Phi_{1/\zeta}, \ldots, \Phi_{R-1/\zeta} \tag{1.4}$$

are functions of a and β which are independent of k, f_0 is not constant, Φ_0 is not identically zero, and Φ_R is a function of (a, β, k).

Subsequently, the function f_0 will be called the principal part of the change function, the functions $f_{\chi+\lambda}$ will be referred to as the coefficients of the expansion of the change function, $\Phi_u (U < R)$ will be known as the coefficients of the expansion of the intensity function, and Φ_R will be called the remainder term.

The number k in the formula (1.3) is related to h by the formula (1.2). For a fixed h, the rate of change of the function Φ increases with an increase in t. We will therefore call t the change index of the considered integral. Our first problem consists in the investigation of the properties of the integrals of the form (1.3) in so far as they may depend on the change index t.

3. The operators L and N have the same structure as had the operator L in article [1]. Formula (1.9) of that paper is valid for L and N, and we may write:

$$L(\Phi)=e^{kf}\left\{k^{l}\sum_{v=0}^{v=l}k^{-v}\sum_{u=0}^{u=R}{}^{*}k^{-u}L_{v}(\Phi_{u})\right\}$$
$$N(\Phi)=e^{kf}\left\{k^{n}\sum_{q=0}^{q=n}k^{-q}\sum_{p=0}^{p=R}{}^{*}k^{-p}N_{q}(\Phi_{p})\right\} \tag{1.5}$$

In these formulas L_v and N_q are differential operators of orders v and q respectively. Their coefficients are polynomials in the derivatives of f of degree $l - v$, and $n - q$, respectively. Taking into consideration the expression (1.3) for f, we may write

$$L_v = L_{v,0}+\sum_{\lambda=0}^{\lambda=\lambda'}k^{-\frac{\varkappa+\lambda}{\zeta}}L_{v,(\varkappa+\lambda)/\zeta}$$
$$N_q = N_{q,0}+\sum_{\mu=0}^{\mu=\mu'}k^{-\frac{\varkappa+\mu}{\zeta}}N_{q,(\varkappa+\mu)/\zeta} \tag{1.6}$$

where L and N with two subscripts no longer depend on k,

$$\lambda'=(l-v)(\zeta-\varkappa-1),\qquad \mu'=(n-q)(\zeta-\varkappa-1)$$

Substituting (1.6) into (1.5), after some transformations we obtain

$$L(\Phi)=e^{kf}\left\{k^{l}\sum_{r=0}^{r=l+R}{}^{*}\sum_{u=0}^{u=r}{}^{*}k^{-r}L_{[r-u]}(\Phi_{u})\right\}\qquad (r-u\leqslant l,\ u\leqslant R)$$
$$N(\Phi)=e^{kf}\left\{k^{n}\sum_{s=0}^{s=n+R}{}^{*}\sum_{p=0}^{p=s}{}^{*}k^{-s}N_{[s-p]}(\Phi_{p})\right\}\qquad (s-p\leqslant n,\ p\leqslant R) \tag{1.7}$$

In these, and in all subsequent formulas, we should retain in the sums only those terms in which the summation indices satisfy the inequalities indicated in parentheses. The expressions $L_{[w]}$ and $N_{[w]}$ are given by

$$L_{w}=\sum_{r+\gamma/\zeta=w}L_{r,\gamma/\zeta},\qquad N_{[w]}=\sum_{r+\gamma/\zeta=w}N_{r,\gamma/\zeta}$$

(the summation being carried out over all nonnegative integers r and γ, for which $r+\gamma/\zeta=w$).

We can show that $L_{[w]}$ has the following properties (assuming that w' stands for the integer part of w).

(a) $L_{[w]}$ is a differential operator of order w' whose coefficients are polynomials in terms of the functions f_0, f_χ, $f_{\chi+1}$, $\ldots$, $f_{\zeta-1}$ and their derivatives, and linear functions of the $a_{jk}{}^{(\nu)}$.

(b) When w is an integer, the principal part of $L_{[w]}$ coincides with the principal part of the operator $L_w|_{f=f_0}$ and, in particular, in accordance with formulas (1.7) and (1.8) of the earlier work [1], we have

$$L_{[0]}=\sum_{j=0}^{j=l} a_{j,\,l-j}^{(l)} f_{0\alpha}^{\,j} f_{0\beta}^{\,l-j} \tag{1.8}$$

$$L_{[1]}=\frac{\partial}{\partial f_{0\alpha}}\{L_{[0]}\}\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial f_{0\beta}}\{L_{[0]}\}\frac{\partial}{\partial\beta}+\ldots \tag{1.9}$$

where $f_{0\alpha}$, $f_{0\beta}$ are the derivatives of f_0 with respect to α and β; the succession of dots indicates terms which do not contain symbols of differentiation.

(c) The following formula holds good:

$$L_{[w]}\equiv 0, \qquad 0<w<\frac{\varkappa}{\zeta}$$

$$L_{[\varkappa/\zeta]}=\frac{\partial}{\partial f_{0\alpha}}\{L_{[0]}\}\frac{\partial f_\varkappa}{\partial\alpha}+\frac{\partial}{\partial f_{0\beta}}\{L_{[0]}\}\frac{\partial f_\varkappa}{\partial\beta} \tag{1.10}$$

$$L_{[(\varkappa+\lambda)/\zeta]}=\frac{\partial}{\partial f_{0\alpha}}\{L_{[0]}\}\frac{\partial f_{\varkappa+\lambda}}{\partial\alpha}+\frac{\partial}{\partial f_{0\beta}}\{L_{[0]}\}\frac{\partial f_{\varkappa+\lambda}}{\partial\beta}+F(f_{<\varkappa+\lambda})$$

$$(\lambda=1,\ldots,\zeta-\varkappa-1)$$

Here and subsequently, the symbol $F(f<\chi+\lambda)$ stands for a function of f_0, f_χ, $f_{\chi+1}$, $\ldots$, $f_{\chi+\lambda-1}$ and of their derivatives.

The function $N_{[w]}$ has completely analogous properties.

4. Let us substitute (1.7) into the original equation, replacing h by k with the aid of (1.2), and let us drop the exponential factor. We then obtain

$$k^{n-1/t}\left\{\sum_{s=0}^{s=n+R}{}^{*}\sum_{p=0}^{p=s}{}^{*}k^{-s}N_{[s-p]}(\Phi_p)\right\}+k^l\left\{\sum_{r=0}^{r=l+R}{}^{*}\sum_{u=0}^{u=r}{}^{*}k^{-r}L_{[r-u]}(\Phi_u)\right\}=0 \tag{1.11}$$

$$(r-u\leqslant l,\ u\leqslant R,\ s-p\leqslant n,\ p\leqslant R)$$

Next, following the same procedure as that used in the earlier work [1], we equate to zero the coefficients of the different powers of k beginning with the highest power. Three different cases can arise in this connection.

Case 1.

$$l>n-1/t \quad \text{or} \quad t<\frac{1}{n-l}=t_0$$

Now, assuming that $l = n - 1/t + \rho$, and equating to zero the coefficients of all powers of k from l to $l - R - 1 + 1/\zeta$, we obtain

$$\sum_{u=0}^{u=r}{}^{*} L_{[r-u]}(\Phi_u) = 0 \qquad (r-u \leqslant l;\ u \leqslant R;\ r = 0,\ 1/\zeta, \ldots, \rho - 1/\zeta) \tag{1.12.1}$$

$$\sum_{u=0}^{u=r}{}^{*} L_{[r-u]}(\Phi_u) + \sum_{p=0}^{p=r-\rho}{}^{*} N_{[r-\rho-p]}(\Phi_p) = 0 \tag{1.12.2}$$

$-u \leqslant l;\ u \leqslant R;\ r-\rho-p \leqslant n;\ \ p \leqslant R;\ r = \rho,\ \rho + 1/\zeta,\ \rho + 2/\zeta, \ldots, R + 1 - 1/\zeta)$

$$\sum_{r=R+1}^{r=l+R}{}^{*} \sum_{u=0}^{u=R}{}^{*} k^{-r} L_{[r-u]}(\Phi_u) + \sum_{s=R+1}^{s=n+R+\rho}{}^{*} \sum_{p=0}^{p=s-\rho}{}^{*} k^{-s} N_{[s-\rho-p]}(\Phi_p) = 0 \tag{1.12.3}$$

$$(r-u \leqslant l,\ s-\rho-p \leqslant n,\ p \leqslant R)$$

Case 2.

$$l < n - 1/t \quad \text{or} \quad t > \frac{1}{n-l} = t_0$$

Now, assuming that $n - 1/t = l + \rho$, we similarly obtain the following equations

$$\sum_{p=0}^{p=s}{}^{*} N_{[s-p]}(\Phi_p) = 0 \qquad (s-p \leqslant n;\ p \leqslant R;\ s = 0,\ 1/\zeta,\ 2/\zeta, \ldots, \rho - 1/\zeta) \tag{1.13.1}$$

$$\sum_{p=0}^{p=s}{}^{*} N_{[s-p]}(\Phi_p) + \sum_{u=0}^{u=s-\rho}{}^{*} L_{[s-\rho-u]}(\Phi_u) = 0 \tag{1.13.2}$$

$(s-p \leqslant n;\ p \leqslant R;\ s-\rho-u \leqslant l,\ u \leqslant R;\ s = \rho,\ \ \rho + 1/\zeta,\ \ \rho + 2/\zeta, \ldots, R + 1 - 1/\zeta)$

$$\sum_{s=R+1}^{s=n+R}{}^{*} \sum_{p=0}^{p=R}{}^{*} k^{-s} N_{[s-p]}(\Phi_p) + \sum_{r=R+1}^{r=l+R+\rho}{}^{*} \sum_{u=0}^{u=r-\rho}{}^{*} k^{-r} L_{[r-\rho-u]}(\Phi_u) = 0 \tag{1.13.3}$$

$$(s-p \leqslant n;\ r-\rho-u \leqslant l;\ u \leqslant R)$$

Case 3.

$$l = n - 1/t \quad \text{or} \quad t = \frac{1}{n-l} = t_0$$

In this case we obtain

$$\sum_{u=0}^{u=r}{}^{*} L_{[r-u]}(\Phi_u) + \sum_{p=0}^{p=s}{}^{*} N_{[s-p]}(\Phi_p) = 0 \tag{1.14.1}$$

$(u \leqslant R;\quad r-u \leqslant l;\quad p \leqslant R;\quad s-p \leqslant n;\quad r,\ s = 0,\ 1/\zeta,\ 2/\zeta, \ldots, R + 1 - 1/\zeta)$

$$\sum_{r=R+1}^{r=l+R}{}^{*} \sum_{u=0}^{u=R}{}^{*} k^{-r} L_{[r-u]}(\Phi_u) + \sum_{s=R+1}^{s=n+R}{}^{*} \sum_{p=0}^{p=R}{}^{*} k^{-s} N_{[s-p]}(\Phi_p) = 0 \tag{1.14.2}$$

$$(r-u \leqslant l;\ s-p \leqslant n)$$

5. Let us rewrite the lead-off equations of the systems (1.12), (1.13) and (1.14). After dropping the nonzero factors Φ_0, we may write these equations as follows:

$$L_{[0]} \equiv \sum_{j=0}^{j=l} a_{j,\,l-j}^{(l)} f_{0\alpha}^{j} f_{0\beta}^{l-j} = 0 \tag{1.15}$$

$$N_{[0]} \equiv \sum_{j=0}^{j=n} b_{j,\,n-j}^{(n)} f_{0\alpha}^{j} f_{0\beta}^{n-j} = 0 \tag{1.16}$$

$$L_{[0]} + N_{[0]} \equiv \sum_{j=0}^{j=l} a_{j,\,l-j}^{(l)} f_{0\alpha}{}^{j} f_{0\beta}^{l-j} + \sum_{j=0}^{j=n} b_{j,\,n-j}^{(n)} f_{0\alpha}{}^{j} f_{0\beta}^{n-j} = 0 \tag{1.17}$$

Since $L_{[0]}$ and $N_{[0]}$ are characteristic polynomials of the operators L and N respectively, it follows from (1.15) to (1.17) that if integrals of type (1.3) exist, then the level lines of the principal part of the change function f_0* will behave as follows:

(a) when $t < t_0$ they coincide with one of the family of characteristics of the operator L;

(b) when $t > t_0$, they coincide with one of the families of characteristics of the operator N;

(c) when $t = t_0$, they pass along lines which either are not characteristics of L or of N, or else are characteristics of both.

Remark: The trivial solution f_0 = const of the equations (1.15) to (1.17) is not considered.

Integrals of type (1.3), for which the level lines of the principal part of the change function coincide with some family of the real or imaginary curves, will be called integrals corresponding to this family. Thus we can assert that when $t < t_0$ we obtain integrals corresponding to families of characteristics of the operator L, but when $t > t_0$, then the integrals correspond to families of characteristics of the operator N. Integrals of one or other of these types will be called fundamental integrals (or solutions) of equation (1.1).

2. 1. Let us take $t > t_0$, i.e. we consider Case 2, and examine system (1.13) more closely.

This system has a meaning only then when

* Just as in the work [1], the integral (1.3) is constructed in general within a complex region, and, hence, the level lines can be imaginary.

$$n = l + 1/t + \rho \tag{2.1}$$

and ρ is a number of the form $\sigma + \tau/\zeta$ (σ, τ are positive integers). In the opposite case, the expression $L[s - \rho - \mu]$ has no meaning. Henceforward we will always assume that $\rho = \tau + \tau/\zeta$; this means, on the basis of (2.1), that we take a rational number for the change index t. (In monograph [2] $1/t$ was assumed to be an integer).

2. Let $\rho < 1$. We will show that in that case a recurrence process for the determination of the function (1.4) can be constructed if one sets $\chi/\zeta = \rho$.

When $\rho = \chi/\zeta$ in equations (1.13.1), the subscripts of N will be less than χ/ζ, and owing to property (c) of operators $L_{[w]}$, $N_{[w]}$, all equations (1.13.1) will be satisfied if we set $N_{[0]} = 0$; i.e. system (1.13.1) is equivalent to equation (1.16).

From system (1.13.2) we pick the equations corresponding to ($s < 1$). These equations constitute a system of algebraic equations because they contain L and N with only such subscripts as are less than one, which according to property (a) do not involve symbols of differentiation (they are operators of zero order).

After some obvious transformations, this system can be reduced to the form

$$N_{[(\varkappa+\lambda)/\zeta]} = -L_{[\lambda/\zeta]} \qquad (\lambda = 0, 1, 2, \ldots, \zeta - \varkappa - 1) \tag{2.2}$$

We can now determine the coefficients of the expansion of the change function by successive solutions of equations

$$\begin{gathered} \frac{\partial}{\partial f_{0\alpha}}\{N_{[0]}\}\frac{\partial f_\varkappa}{\partial\alpha} + \frac{\partial}{\partial f_{0\beta}}\{N_{[0]}\}\frac{\partial f_\varkappa}{\partial\beta} = -L_{[0]} \\ \frac{\partial}{\partial f_{0\alpha}}\{N_{[0]}\}\frac{\partial f_{\varkappa+\lambda}}{\partial\alpha} + \frac{\partial}{\partial f_{0\beta}}\{N_{[0]}\}\frac{\partial f_{\varkappa+\lambda}}{\partial\beta} = -F(f_{<\varkappa+\lambda}) \end{gathered} \tag{2.3}$$

(see property (c) and expressions $L_{[w]}$, $N_{[w]}$).

Equations (1.13.2), corresponding to ($s \geqslant 1$), can now be rewritten as

$$\begin{gathered} N_{[1]}(\Phi_0) + L_{[1-\rho]}\Phi_0 = 0 \\ N_{[1]}(\Phi_{s-1}) + L_{[1-\rho]}(\Phi_{s-1}) = \\ = -\sum_{p=0}^{p=s-1-1/\zeta}{}^{*} N_{[s-p]}(\Phi_p) - \sum_{u=0}^{u=s-1-1/\zeta}{}^{*} L_{[s-\rho-u]}(\Phi_u) = 0 \\ (s - p \leqslant n;\ \ s - \rho - u \leqslant l;\ \ s = 1 + 1/\zeta, \ldots, R + 1 - 1/\zeta) \end{gathered} \tag{2.4.1}$$

or, expanding the left-hand sides by using formulas analogous to those of (1.9):

$$\left[\frac{\partial}{\partial f_{0\alpha}}\{N_{[0]}\}\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial f_{0\beta}}\{N_{[0]}\}\frac{\partial}{\partial\beta}+\cdots\right](\Phi_0)=0$$

$$\left[\frac{\partial}{\partial f_{0\alpha}}\{N_{[0]}\}\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial f_{0\beta}}\{N_{[0]}\}\frac{\partial}{\partial\beta}+\cdots\right](\Phi_{s-1})=$$
$$=-\sum_{p=0}^{p=s-1-1/\zeta}{}^{*}\, N_{[s-p]}(\Phi_p)-\sum_{u=0}^{u=s-1-1/\zeta}{}^{*}\, L_{[s-\rho-u]}(\Phi_u)$$

(On the left-hand side only the principal parts are written out explicitly. The subscripts are subjected to the relations given earlier).

On this basis the coefficients of the expansion of the intensity function can be determined by the method of successive integration of first-order linear equations. For the determination of the remainder term there remains equation (1.13.3), which may be rewritten in the form

$$\sum_{s=R+1}^{s=n+R}{}^{*}\, k^{-s}N_{[s-R]}(\Phi_R)+\sum_{r=R+\theta}^{r=l+R+\rho}{}^{*}\, k^{-r}L_{[r-\rho-R]}(\Phi_R)=F \qquad (2.5)$$

where F is some definite expression in terms of the function (1.4), while θ is the larger one of the numbers 1 and ρ.

3. Next, suppose $\rho \geqslant 1$. We can then assume that the change function is independent of k, that is $f = f(\alpha, \beta) = f_0(\alpha, \beta)$, and that the coefficients Φ_u of the expansion of the change function with integer subscripts are identically zero for $u < \rho$. We thus obtain a recurrence process for determining the functions

$$f_0,\ \Phi_0,\ \Phi_1,\ \Phi_2,\ \ldots,\ \Phi_{\rho'},\ \Phi_\rho,\ \Phi_{\rho+1/\zeta},\ \ldots,\ \Phi_{R-1/\zeta}$$

(ρ' is the largest integer less than ρ). The recurrence process mentioned is described below.

The change function f_0 is determined by means of equation (1.16). Functions Φ_0, Φ_1, ..., $\Phi_{\rho'}$ are determined with the aid of equations

$$N_1(\Phi_{s-1})=-\sum_{p=0}^{p=s-2}N_{s-p}(\Phi_p) \qquad (s-p\leqslant n;\ s=1,\ldots,\theta) \qquad (2.4.2)$$

where θ is the smaller of numbers R and ρ'. Functions Φ_ρ, $\Phi_{\rho+1/\zeta}$, ..., $\Phi_{R-1/\zeta}$ (when $\rho' < R$) are determined by the use of equations

$$N_1(\Phi_{s-1})=-\sum_{p=0}^{p=s-1-1/\zeta}{}^{*}\, N_{s-p}(\Phi_p)-\sum_{u=0}^{u=s-\rho}{}^{*}\, L_{s-\rho-u}(\Phi_u) \qquad (2.4.3)$$
$$(s-p\leqslant n;\quad s=\rho,\quad \rho+1/\zeta,\ldots,R+1-1/\zeta)$$

The remainder term is determined from equation (2.5), just as when $\rho < 1$.

Remarks.

(a) If $\rho \geqslant 1$, and all the functions $f_{\chi+\lambda}$ are assumed to be identically zero, $L_{[w]}$ and $N_{[w]}$ do not differ from L_w and N_w respectively.

(b) If ρ is an integer, it may be assumed that all Φ_u with fractional subscripts are zero, i.e. an integral of the same type as in the earlier work [1] is sought.

(c) If $\rho < 1$, the recurrence process for the determination of the functions (1.4) can be obtained by setting $\chi/\zeta = \rho/m$ (m is an integer); just as when $\rho \geqslant 1$, we may set $\chi/\zeta = 1/m$. The processes described for the construction of functions (1.4) are not unique. They are only the simplest, and lead to integrals sufficiently general for finding the solutions of problems of special interest.

4. The coefficients of the original equation are assumed to be sufficiently smooth. It is for this reason that the only singular points of the equation (1.16), (2.3), (2.4) and (2.5) will be those where the coefficients of the highest-order derivatives vanish simultaneously.

For the equations (1.16) and (2.5), these will be the points at which all the $b_{ij}^{(n)}$ vanish simultaneously, i.e. the singular points of operator N.

For the equations (2.3) and (2.4), these can be only those points at which the following equation holds

$$\frac{\partial}{\partial f_{0\alpha}}\{N_{[0]}\} = \frac{\partial}{\partial f_{0\beta}}\{N_{[0]}\} = 0 \tag{2.6}$$

i.e. (a) singular points of operator N; (b) stationary points of function f_0; (c) points at which the characteristics of N are repeated, or points of common tangency between N characteristics belonging to different families.

The trivial solutions f_0 = const are disregarded. The case when all the $b_{ij}^{(n)}$ are identically zero in the region in question is also excluded from consideration without loss of generality. We may therefore conclude that all the points of a region G cannot be singular points of the equations (1.16) and (2.5). For the equations (2.3) and (2.4) this can happen only then when the operator N has multiple families of characteristics in the region G.

5. In the equations that determine f_χ (when $\rho < 1$) or $\Phi_{\rho-1}$ (when $\rho \geqslant 1$), the term which depends on the operator L has a coefficient $L_{[0]}$

which can vanish identically only when the regions of L and N being considered have coinciding families of characteristics. If this case is excluded, we may assert that the terms in (2.4) depending on operator L will enter into the computations from a certain stage, when we begin to determine $\Phi_{\rho-1}$. Thus, we say that if $t > t_0$, then the integral of type (1.3) can be determined (when $\rho \geqslant 1$) from the approximating equation

$$hN(\Phi) = 0 \tag{2.7}$$

to within the asymptotic error of the order $k^{-\rho+1}$. This means that for every problem for which $R = \rho - 1$, the remainder term $\Phi_R = \Phi_{\rho-1}$ will be bounded, that the replacement of equation (1.1) by equation (2.7) will not affect the change function, and that the error caused by this replacement will be of the order $O(k^{-\rho+1})$ in the determination of the intensity function. If $\rho < 1$, the equation (2.7) is not applicable to the determination of the intensity function, and it will also give an error in the determination of the change function.

The principal part of the change function can however, be constructed precisely for an arbitrary ρ.

If $L_{[0]} = 0$, then the asymptotic error of the equation (2.7) decreases, but we cannot go into greater detail on this question here.

3. Let $t < t_0$, i.e. we have case 1. We must then use system (1.12) to determine the coefficients of the expansion of the change function and of the intensity function. This system (1.12) differs from system (1.13) only in that L is replaced by N, and N by L. Therefore, by analogy with Section 2, we can formulate the final results at once without having to make any explanations.

2. Let the following relation be given

$$l = n - 1/t + \rho$$

(as before, ρ is a number of the form $\sigma + r/\zeta$).

If $\rho < 1$, we set $\chi/\zeta = \rho$, and obtain:

(a) to determine the principal part of the change function, equation (1.15);

(b) to determine the coefficients of the expansion of the change functions, equations

$$\frac{\partial}{\partial f_{0\alpha}}\{L_{[0]}\}\frac{\partial f_\varkappa}{\partial \alpha} + \frac{\partial}{\partial f_{0\beta}}\{L_{[0]}\}\frac{\partial f_\varkappa}{\partial \beta} = -N_{[0]} \tag{3.1}$$

$$\frac{\partial}{\partial f_{0\alpha}}\{L_{[0]}\}\frac{\partial f_{\varkappa+\lambda}}{\partial \alpha}+\frac{\partial}{\partial f_{0\beta}}\{L_{[0]}\}\frac{\partial f_{\varkappa+\lambda}}{\partial \beta}=-F(f_{<\varkappa+\lambda}) \tag{3.2}$$

$$(\lambda = 1, 2, \ldots, \zeta - \varkappa - 1)$$

(c) to determine the coefficients of the expansion of the intensity function, equations

$$\left[\frac{\partial}{\partial f_{0\alpha}}\{L_{[0]}\}\frac{\partial}{\partial \alpha}+\frac{\partial}{\partial f_{0\beta}}\{L_{[0]}\}\frac{\partial}{\partial \beta}+\ldots\right](\Phi_{r-1}) =$$

$$= -\sum_{u=0}^{u=r-1-1/\zeta}{}^{*} L_{[r-u]}(\Phi_u) - \sum_{p=0}^{p=r-1-1/\zeta}{}^{*} N_{[r-p-\rho]}(\Phi_p) \tag{3.3.1}$$

$$(r-u \leqslant l;\ r-p-\rho \leqslant n;\ r = 1, 1+1/\zeta, \ldots, R+1-1/\zeta)$$

(d) to determine the remainder term, equation

$$\sum_{r=R+1}^{r=l+R}{}^{*} k^{-r} L_{[r-R]}(\Phi_R) + \sum_{s=R+\theta}^{s=n+R+\rho}{}^{*} k^{-s} N_{[s-\rho-R]}(\Phi_R) = F \tag{3.4}$$

where F is some definite expression involving functions (1.4), and θ is the larger of the two numbers 1 and ρ.

3. If $\rho \geqslant 1$, then we may assume that $f = f_0$ and Φ_u is different from zero only for integer values of $u < \rho$; in that case we have the following results:

(a) f_0 is again determined by means of equation (1.15);

(b) functions $\Phi_u(u < \rho)$ are determined by equation

$$L_1(\Phi_{r-1}) = -\sum_{u=0}^{u=r-2} L_{r-u}(\Phi_u) \qquad (r-u \leqslant l;\ \ r = 1, \ldots, \theta) \tag{3.3.2}$$

(θ is the smaller of the numbers R and ρ', while ρ' is the largest integer less than ρ);

(c) the functions $\Phi_u(u \geqslant \rho)$ are determined, when $\rho' < R$, by equations

$$L_1(\Phi_{r-1}) = -\sum_{u=0}^{u=r-1-1/\zeta}{}^{*} L_{r-u}(\Phi_u) - \sum_{p=0}^{p=r-\rho}{}^{*} N_{r-p-\rho}(\Phi_p)$$

$$(r-u \leqslant n;\quad r-p-\rho \leqslant n:\ \ r = \rho+1,\ \ \rho+1+1/\zeta, \ldots, R+1-1/\zeta)$$

(d) the remainder term Φ_R is determined by means of equation (3.4).

4. The only singular points the equations determining functions (1.4) and the remainder term Φ_R can have are the singular points of operator

L, the stationary points of f_0, the multiple points of the characteristics of L and the common tangency points of characteristics of L belonging to different families.

5. If, as before, we assume L and N to have no overlapping families of characteristics in the region being considered, then $N_{[0]} \not\equiv 0$, and the quantities connected with operator N will enter into our derivations only when we come to determining function $\Phi_{\rho-1}$. Hence for $t < t_0$, an integral of type (1.3) can be determined from the approximate equation

$$L(\Phi) = 0 \tag{3.5}$$

with an asymptotic error of order $k^{-\rho+1}$. This statement must be understood in the same sense as in Section 2.

4. 1. Let us assume that $t = t_0$, i.e. we have case 3 and consider system (1.14). In this case the recurrence process to determine functions (1.4) can be obtained for an arbitrary rational value $\chi/\zeta < 1$. (In particular, we may set $\chi/\zeta = 0$, i.e. take $f = f_0$; but this will not yield integrals general enough for solving the problems we are to consider.) We will describe the process corresponding to the case when $\chi/\zeta \neq 0$. In doing so, we will not dwell on the consideration of case $\rho < 1$, for such considerations would be identical with those in Section 2.

(a) To determine f_0 we have equation (1.17).

(b) The coefficients of the expansion of the change function are determined by means of equations

$$\frac{\partial}{\partial f_{0\alpha}}\{L_{[0]} + N_{[0]}\}\frac{\partial f_\varkappa}{\partial \alpha} + \frac{\partial}{\partial f_{0\beta}}\{L_{[0]} + N_{[0]}\}\frac{\partial f_\varkappa}{\partial \beta} = 0$$

$$\frac{\partial}{\partial f_{0\alpha}}\{L_{[0]} + N_{[0]}\}\frac{\partial f_{\varkappa+\lambda}}{\partial \alpha} + \frac{\partial}{\partial f_{0\beta}}\{L_{[0]} + N_{[0]}\}\frac{\partial f_{\varkappa+\lambda}}{\partial \beta} + \ldots + F(f_{<\varkappa+\lambda}) = 0 \tag{4.1}$$

$$(\lambda = 1, 2, \ldots, \zeta - \varkappa - 1)$$

(c) The coefficients of the expansion of the intensity function can be determined successively with equations

$$\left[\frac{\partial}{\partial f_{0\alpha}}\{L_{[0]} + N_{[0]}\}\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial f_{0\beta}}\{L_{[0]} + N_{[0]}\}\frac{\partial}{\partial \beta} + \ldots\right](\Phi_0) = 0$$

$$\left[\frac{\partial}{\partial f_{0\alpha}}\{L_{[0]} + N_{[0]}\}\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial f_{0\beta}}\{L_{[0]} + N_{[0]}\}\frac{\partial}{\partial \beta} + \ldots\right](\Phi_{r-1}) =$$

$$\doteq -\sum_{u=0}^{u=r-1-1/\zeta}{}^{*} L_{[r-u]}(\Phi_u) - \sum_{p=0}^{p=s-1-1/\zeta}{}^{*} N_{[s-p]}(\Phi_p) \tag{4.2}$$

$$(r - u \leqslant l;\ s - p \leqslant n;\ r, s = 1 + 1/\zeta,\ 1 + 2/\zeta, \ldots, R + 1 - 1/\zeta)$$

The remainder term is determined by the use of equations

$$\sum_{r=R+1}^{r=l+R}{}^{*} k^{-r}L_{[r-R]}(\Phi_R) + \sum_{s=R+1}^{s=n+R}{}^{*} k^{-s}N_{[s-R]}(\Phi_R) = \\ = -\sum_{r=R+1}^{r=l+R}{}^{*} \sum_{u=0}^{u=R-1/\zeta}{}^{*} k^{-r} L_{[r-u]}(\Phi_u) - \sum_{s=R+1}^{s=n+R}{}^{*} \sum_{p=0}^{p=R-1/\zeta}{}^{*} k^{-s} N_{[s-p]}(\Phi_p) \\ (r-u \leqslant l;\ s-p \leqslant n) \tag{4.3}$$

2. We will seek such solutions of equation (1.7) as take on zero values on a given real contour γ not touching the characteristics of L or N.

Let contour γ be made to coincide (if necessary by means of a preliminary real transformation of the independent variables) with the line $\alpha = \alpha_0$. Then $f_0 = 0$ on γ, and (1.7) yields

$$a_{l,0}^{(l)} f_{0\alpha}^{\ l} + b_{n,0}^{(n)} f_{0\alpha}^{\ n} = 0 \quad \text{on } \gamma \tag{4.4}$$

The functions $a_{l,0}^{(l)}$ and $b_{n,0}^{(n)}$ are distinct from zero at all points of γ, for otherwise the characteristics of L or N would touch the contour γ. Therefore, the equation (4.4) yields $n - l$ non-zero values for $f_{0\alpha}$ on γ:

$$f_{0\alpha} = \nu \sqrt[n-l]{\left|\frac{a_{l,0}^{(l)}}{b_{n,0}^{(n)}}\right|} \quad \text{on } \gamma \tag{4.5}$$

where ν is a root of the equation

$$\nu^{n-l} + \operatorname{sign}\left(\frac{a_{l,0}^{(l)}}{b_{n,0}^{(n)}}\right) = 0 \tag{4.6}$$

Differentiating (1.7) with respect to α, and setting $\alpha = \alpha_0$ in the result, equations can be constructed for the successive determination of the contour values of any order derivative of f_0 with respect to α. These equations have the form

$$\frac{\partial}{\partial f_{0\alpha}}[a_{l,0}^{(l)} f_{0\alpha}^{\ l} + b_{n,0}^{(n)} f_{0\alpha}^{\ n}] \frac{\partial^r f_0}{\partial \alpha^r} = F\left(\frac{\partial f_0}{\partial \alpha}, \frac{\partial^2 f_0}{\partial \alpha^2}, \ldots \frac{\partial^{r-1} f_0}{\partial \alpha^{r-1}}\right) \text{ on } \gamma \tag{4.7}$$

where F is a known function of the arguments enumerated.

The coefficient of the r-th order derivative in the left hand side is obviously different from zero, for $f_{0\alpha}$ stands for a nonzero root of equation (4.4), and the latter, being a root of the binomial, cannot be a multiple root. Hence, to every solution of (4.5) there corresponds some integral of equation (1.17), which on some obvious hypotheses can be determined in the neighborhood of γ by means of Taylor's series.

3. If f_0 is known, then the remaining functions (1.4) and the remainder term are obtained by integrating the equations (4.1), (4.2) and (4.3). Thus, to each of the solutions of (4.5) there corresponds a certain class of integrals of type (1.3). As in the earlier work [2], we call these "integrals with the support contour γ". From this it follows that these are the solutions of equation (1.1) for which the principal part of the change function vanishes on γ.

We will show that equations (4.1), (4.2) and (4.3) have no singular points on γ. By hypothesis, $b_{n,0}^{(n)} \neq 0$ on γ. Therefore, there are no singular points of operator N on γ, and hence equation (4.3) has no singular points. Singular points of equations (4.1) and (4.2) can occur only where the following equation holds good:

$$\frac{\partial}{\partial f_{\alpha 0}} \{L_{[0]} + N_{[0]}\} = \frac{\partial}{\partial f_{0\beta}} \{L_{[0]} + N_{[0]}\} = 0$$

and since the contour value of $f_{0\alpha}$ is zero, we must have the relation

$$\frac{\partial}{\partial f_{0\alpha}} \{L_{[0]} + N_{[0]}\} = \frac{\partial}{\partial f_{0\alpha}} \{a_{l,0}{}^{(l)} f_{0\alpha}{}^{l} + b_{n,0}{}^{(n)} f_{0\alpha}{}^{n}\} = 0 \quad \text{on} \quad \gamma$$

but this contradicts a result established above .

The contour value $f_{0\alpha}$ in the integrals with support contour γ is proportional to the corresponding root of binomial equation (4.6).

Generally speaking, these roots can be separated into two sets: $(n - l)/2$ of them have positive real parts, $(n - l)/2$ of them have negative real parts. The exception represents cases when $n - l$ is an odd integer or when equation (4.6) has two pure imaginary roots.

5. 1. Let us consider a problem analogous to problem A in the earlier article [1]. In the finite simply-connected region $\overline{\Gamma} = \Gamma + \gamma$, bounded by contour γ, parameters (α, β) correspond to a system of coordinates similar to a polar system, i.e. contour γ is given by equation $\alpha = \alpha_0 > 0$, and the region is determined by the inequalities

$$0 \leqslant \alpha \leqslant \alpha_0, \qquad 0 \leqslant \beta \leqslant 2\pi$$

The correspondence between the points of the region and the number pair (α, β) is one-to-one everywhere except at points $\alpha = 0$ and lines $\beta = 0$, $\beta = 2\pi$.

Problem A consists in constructing the solution of equation (1.1) in the region Γ under the boundary condition

$$\frac{\partial^{\mu}\Phi}{\partial\alpha^{\mu}} \equiv D^{(\mu,0)}(\Phi) = k^{\mu} g^{(\mu)} e^{ik\varphi} \qquad \text{on} \quad \gamma \tag{5.1}$$

$$\left(\mu = 0,1,2,\ldots, \frac{n}{2} - 1\right) \quad (n - \text{ even})$$

where $g^{(\mu)}$ and ϕ are given functions of β independent of k, while ϕ is assumed to be a real function. Just as in the earlier work [1], the parameters of the problem are assumed to be sufficiently smooth, i.e. γ is a sufficiently smooth contour, while $g^{(\mu)}$ and $e^{ik\phi}$ are sufficiently smooth functions of points on contour γ. Moreover, it is assumed that ϕ' does not vanish at any point of γ. Our aim will be to show that the approximate solution of problem A under known conditions will consist either of fundamental integrals only or of fundamental integrals and integrals with support contours γ (where γ is the boundary of the region).

2. As will be shown later, the solution of problem A can be sought by assuming the parameters t in boundary conditions (5.1) and in formula (1.3) to have the same meaning. Therefore, in the concretely formulated problem A, parameter k is determined by the character of the boundary functions, and it is natural to call it the index of changeability [or variability] of problem A.

3. Let us first consider the case when $t > t_0$, and assume N to be an elliptic operator without singular points of γ, all whose families of characteristics are simple near γ. Moreover, we will assume that on γ there are no points of common tangency of characteristics of N belonging to different families.

In this case we seek a solution of problem A in the form

$$\Phi = \Sigma\, \Phi_*{}^{(q)} e^{kf^{(q)}} \tag{5.2}$$

where $f^{(q)}$ and $\Phi_*{}^{(q)}$ stand for the change function and intensity function of the fundamental integral corresponding to the q-th family of the characteristics of N. The summation is carried out over a certain number of families of characteristics of N, to be specified later.

4. Functions $f_0{}^{(q)}$ and $f_{\chi+\lambda}{}^{(q)}$ will be subjected to the following conditions

$$f_0{}^{(q)} = i\varphi(\beta) \quad \text{on} \quad \gamma \tag{5.3.1}$$

$$\mathrm{Re}\,\{f_{0\alpha}{}^{(q)}\} > 0 \quad \text{on} \quad \gamma \tag{5.3.2}$$

$$f_{\varkappa+\lambda}{}^{(q)} = 0 \quad \text{on} \quad \gamma \tag{5.4}$$

Condition (5.3.2) is called the "damping condition" in the earlier work [1]. It was there shown that only those integrals can be subjected to this condition which correspond to certain definite families of characteristics, in which there will be $n/2$ in an elliptic operator of order n. In sum (5.2) we will therefore retain only the $n/2$ terms corresponding to these $n/2$ families of characteristics.

The principal part of the change function f_0 has no stationary points on γ, for by hypothesis $\phi'(\beta)$ is never equal to zero; the singular points of N, and the points of common tangency of the characteristics of N belonging to different families, are absent from γ. This means that the equation which determines f_0, $f_{\chi+\lambda}$ and Φ_u has no singular points on γ, and we assume that in the neighborhood of γ, the functions $f_0^{(q)}$ and $f_{\chi+\lambda}^{(q)}$ satisfying the contour conditions (5.3) and (5.4) will be uniformly bounded in α, β. Then, for sufficiently large k, the function

$$f^{(q)} = f_0^{(q)} + \sum_{\lambda=0}^{\lambda=\zeta-\varkappa-1} k^{-\frac{\varkappa+\lambda}{\zeta}} f_{\varkappa+\lambda}$$

will satisfy the following condition

$$f^{(q)} = i\varphi \tag{5.5.1}$$

$$\operatorname{Re}\{f_\alpha^{(q)}\} > 0 \quad \text{on} \quad \gamma \tag{5.5.2}$$

as in the earlier work [1].

5. Having replaced L by $D^{(\mu,9)}$ and Φ and f by $\Phi^{(q)}$ and $f^{(q)}$ in formula (1.5), we obtain

$$D^{(\mu,0)}(\Phi^{(q)}) = e^{kf^{(q)}} \{k^\mu \sum_{v=0}^{v=\mu} k^{-v} \sum_{u=0}^{u=R}{}^{*} k^{-u} D_{v,q}^{(\mu,0)}(\Phi_u^{(q)})\}$$

Let us sum the expression obtained over $n/2$ values of q, substitute conditions (5.1) and divide out the exponential factor. We thus obtain the contour relation

$$\sum_{q=1}^{q=\frac{1}{2}n} k^\mu \sum_{v=0}^{v=\mu} k^{-v} \sum_{u=0}^{u=R}{}^{*} k^{-u} D_{v,q}^{(\mu,0)}(\Phi_u^{(q)}) = k^\mu g^{(\mu)} \quad \text{on } \gamma \tag{5.6}$$

which is entirely analogous to contour relation (2.3) obtained in the earlier work [1] in solving problem A for an equation which did not contain a small parameter.

6. Henceforward the solution of problem A is constructed exactly as in the earlier work [1].

In (5.6) we equate the coefficients of corresponding powers of k from μ to $\mu - R + 1/\zeta$ on both sides of the equation. As was shown in [1], this makes it possible to attach to each differential equation (2.4) the contour condition which consists in specifying the values of function $\Phi_u^{(q)}$

on γ. We assume proposition (a) that there exists a small enough neighborhood of γ in which all the functions $\Phi_u (u < R)$ so determined are uniformly bounded in α, β.

For the remaining terms, from (5.6) we obtain $n/2$ contour conditions similar to conditions (2.6) in article [1]. The problem on the construction of functions $\Phi_R{}^{(q)}$ in the neighborhood of γ thus remains indeterminate, since every one of functions $\Phi_R{}^{(q)}$ satisfies an equation of order n. We next assume proposition (b), that this problem can be made determinate by specifying additional contour conditions so that functions $\Phi_R{}^{(q)}$ may be uniformly bounded in α, β, and k in some neighborhood of γ.

Let us consider the expression

$$\Phi' = \psi \sum_{q=1}^{q=n/2} \Phi^{(q)} + \Phi^{(0)} \qquad \left(\Phi^{(q)} = e^{kf^{(q)}} \sum_{u=0}^{u=R}{}^{*} k^{-u} \Phi_u\right)$$

Here ψ is a smoothing function which has the same meaning as in the earlier work [1]. It is equal to one in region Γ_η, i.e. when $\alpha_0 \geqslant \alpha \geqslant \alpha_0 - \eta$, equal to zero in region $\Gamma - \Gamma_\epsilon$, i.e. when $\alpha_0 - \epsilon > \alpha \geqslant 0$ $(0 < \eta < \epsilon < \alpha_0)$, and is unboundedly differentiable in region $\Gamma_\epsilon - \Gamma_\eta$, i.e. when $\alpha_0 - \eta \geqslant \alpha \geqslant \alpha_0 - \epsilon$.

Let us require $\Phi^{(0)}$ to satisfy equation (1.1). We then obtain the following equation to determine Φ:

$$hN(\Phi^{(0)}) + L(\Phi^{(0)}) = P \equiv -(hN + L)\left(\psi \sum_{q=1}^{q=n/2} \Phi^{(q)}\right)$$

We select ϵ so small that propositions (a) and (b) hold good in Γ_ϵ, and that the damping condition (5.5) is also satisfied. The absolute value P will then be of type $O(k^{-v})$, where v is arbitrary. In region Γ_η, contained in Γ_ϵ, all $\Phi^{(q)}$ satisfy equation (1.1), while $\psi = 1$, and hence $P = 0$. In region $\Gamma - \Gamma_\epsilon$, P is also zero since $\chi = 0$. In $\Gamma_\epsilon - \Gamma_\eta$, the absolute values of $\Phi^{(q)}$ are of type $O(k^{-v})$ because of equation

$$\Phi^{(q)} = e^{kf^{(q)}} \sum_{u=0}^{u=R}{}^{*} k^{-u} \Phi_u{}^{(q)}$$

and owing to propositions (a) and (b), while ψ and its derivatives are bounded. Therefore, the absolute value of P is of type $O(k^{-v})$.

Introducing proposition (c) that the nonhomogeneous form of equation (1.1) has a bounded solution for an arbitrary bounded right hand-side and for homogeneous condition (5.1), we may assert that the approximate solution of problem A, without Φ^0 and the remainder terms, can be found in the form

$$\Phi' = \psi \sum_{q=1}^{q=n/2} e^{kf^{(q)}} \sum_{u=0}^{u=R-1/\zeta}{}^{*} \Phi_u^{(q)}$$

with an error of the order $O(k^{-R})$.

7. Propositions (*a*) and (*c*) pertain to classical problems of the theory of differential equations, and we will not dwell upon them. Proposition (*b*) was considered in the earlier article [1], but certain errors were made in the discussion there.*

The question of making the problem on the Φ_R determinate is more difficult to settle than had been expected. A complete statement of the difficulties is not appropriate here. We will restrict ourselves to the consideration of an example illustrating the matter.

Let the following equation be given in polar coordinates (r, θ):

$$\left[k^{-1}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + a\right) + \frac{2}{r}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right)\right]\Phi_R = g \qquad (5.7)$$

(a, g are given functions, k is a large constant). The following contour condition is also given

$$\Phi_R/_{r=1} = 0 \qquad (5.8)$$

It is required to specify the contour values of the derivative of Φ_R, with respect to r, in such a way that the corresponding Cauchy problem may have a bounded solution (as $k \to \infty$) in the neighborhood of $r = 1$ (for $r < 1$). The question of making the problem on the remainder term determinate in the example considered in [1] (Section 6), is equivalent to the problem just stated.

We introduce the notation

$$\left.\frac{\partial^m \Phi_R}{\partial r^m}\right|_{r=1} = \Phi_m, \qquad \left.\frac{\partial^m g}{\partial r^m}\right|_{r=1} = g_{(m)}$$

and construct the system of equations obtained when (5.7) is differentiated successively with respect to r, and $r = 1$ is set in all but the original one of these equations. Making use of (5.8), we then obtain

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* In particular, in the so-called auxiliary characteristic equation one term was omitted, and the equation actually has the form

$$\{a_{l^*}^{(l)}[(f_\alpha + k^{1-}\lambda)^l - f_\alpha^l]\}_{\alpha=\alpha_0} = 0$$

$$k^{-1}(\Phi_{(2)} + \Phi_{(1)}) + 2\Phi_{(1)} = g_{(0)} \tag{5.9}$$
$$k^{-1}\left(\Phi_{(3)} + \Phi_{(2)} - \Phi_{(1)} + \frac{\partial^2\Phi_{(1)}}{\partial\theta^2} + a\Phi_{(1)}\right) - 2\Phi_{(2)} - 2\Phi_{(1)} + 2i\,\frac{\partial\Phi_{(1)}}{\partial\theta} = g_{(1)}$$
. . . .

Letting

$$\Phi_{(1)} = \sum_{p=0}^{\infty} k^{-p}\varphi_p \qquad (\varphi_p \text{ — are functions of } \theta$$

and applying a familiar procedure, by means of (5.9) we can successively determine the contour values of all the derivatives of Φ_R with respect to r and construct the solution of Cauchy's problem in the form of a Taylor series in r. In doing this, $\Phi_{(m)}$ will in general be represented in terms of infinite series containing positive powers of k, and the boundedness of the solution as $k \to \infty$ will thus not be guaranteed. Therefore, we require that all the $\Phi_{(m)}$ be represented in the form of a Maclaurin series in powers of k^{-1}. This leads us to the equations

$$2\varphi_0 = g_0, \qquad -2\varphi_0 - 4\varphi_1 + 2i\,\frac{\partial\varphi_0}{\partial\theta} = g_1 \ldots$$

from which we can successively determine functions Φ, ϕ_0, ..., and the problem has thus been made determinate.

The method presented of determining the problem on th Φ_R can be carried over to the general case of interest to us (when the coefficients of the equation and the given data of problem A are analytic functions). A detailed discussion of this matter is, however, quite cumbersome and would be out of place here.*

8. Thus, when $t > t_0$, problem A for an equation with a small parameter has the same properties as when the equation does not contain a small parameter.

For large enough k, problem A has a rapidly damped solution, which can be constructed with arbitrary accuracy by the superposition of fundamental integrals with indices of change equal to the change index of the problem. (This assertion is of a conditional nature in the sense that the original equation has to satisfy certain conditions which will ensure the validity of certain propositions assumed in this paper.)

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* This type of method was actually used in the earlier work [1], but there a mistake was made in the auxiliary contour conditions by showing only the principal terms.

The actual construction of the approximate solution of the problem (without the remainder term) reduces to the integration of equations (1.16) with contour conditions (5.3.1), equations (2.3) with contour conditions (5.4) and equations (2.4) for given contour values of function Φ_u. Since the resulting solution is composed of integrals corresponding to $n/2$ families of characteristics of N, equation (1.16) can be replaced by $n/2$ linear first-order differential equations (see [1]). Finally, the approximate solution of problem A is reduced to successively solving Cauchy's problem for first-order linear equations in the neighborhood of γ.

9. If an asymptotic error of order $k^{-\rho+1}$ is allowed, then the solution of problem A can be constructed by means of approximated equation (2.7). Greater accuracy requires the consideration of L, but the quantities connected with this operator can introduce corrections only into the free term of the equation and into the boundary conditions determining the coefficients of expansion of the change and intensity functions.

6. 1. Let $t < t_0$, while L is an elliptic operator with simple (nonmultiple) families of characteristics having no singular points in Γ, and possessing no points of common tangency of characteristics belonging to distinct families. In this case a solution of problem A cannot be constructed by the superposition of fundamental integrals only, for when $t < t_0$ the number (l) of different families of fundamental integrals is less than the required number (n). It becomes necessary to take integrals with support contours into consideration.

2. Let us consider the integral whose supporting contour coincides with region boundary γ. We change the notation used in Section 4 and express the integral in the form:

$$\Phi = \Psi_* e^{\theta g} \tag{6.1}$$

where g and ψ stand for the change function and intensity function, respectively, and θ has the same meaning as k in Section 4, so that

$$g = g_0 + \sum_{\lambda=0}^{\lambda=\xi-\eta-1} \theta^{-\frac{\eta+\lambda}{\xi}} g_{\eta+\lambda}, \qquad \Psi_* = \sum_{u=0}^{u=R} {}^{*}\theta^{-u}\Phi_u \tag{6.2}$$

We will take into consideration only the cases when equation (4.6) has exactly $(n-l)/2$ roots with positive real parts, and we take the solution of problem A in the form

$$\Phi = \sum_{q=1}^{q=\frac{1}{2}l} \Phi_*^{(q)} e^{kf^{(q)}} + \sum_{r=1}^{r=\frac{1}{2}(n-l)} \Psi_*^{(r)} e^{\theta g^{(r)}} \tag{6.3}$$

Here, there are $l/2$ fundamental integrals in the first sum, which correspond to the $l/2$ families of characteristics of L which can be subjected to damping condition (5.3.2), the second sum contains $(n - l)/2$ integrals with the supporting contour γ, which correspond to the roots of equation (4.6) whose real parts are positive.

3. In sum (6.3) and in the integrals with the given supporting contour, the symbols ξ and η are integers, η/ξ is a proper, positive, rational fraction, and θ is determined by the formula

$$\theta = h^{-\frac{1}{n-l}} \tag{6.4}$$

The proper, positive, rational fraction χ/ζ, which appears in the fundamental integrals of the right-hand side of (6.3), is chosen in accordance with the specification given in Section 3, whereby χ and ζ may proportionally change (while still remaining integers). Parameter k of the fundamental integrals is determined by means of formula (1.2).

4. We will express ξ and η in the following forms:

$$\xi = \frac{\zeta}{t(n-l)}, \qquad \eta = \frac{\zeta[1 - t(n-l)]}{t(n-l)} \tag{6.5}$$

The requirement that η/ξ shall be a proper fraction is always fulfilled because of the inequality $0 < t(n - l) < 1$.

Without changing the value of fraction χ/ζ, the integer ζ can be selected in such a way that ξ and η are integers, for t is a rational number. Subsequently ζ will be assumed to be the smallest integer satisfying this requirement. The formulas given above for ξ and η are thus valid. From them, by means of (6.4) and (1.2), we can derive the following equations

$$\theta^{1-\eta/\xi} = k, \qquad \theta^{1/\xi} = k^{1/\zeta}$$

5. The term g_0 is zero on γ in the integrals with given support contour. Making use of the last-displayed formulas, we thus obtain

$$\theta g^{(r)} = \theta^{1-\eta/\xi} \sum_{\lambda=0}^{\lambda=\xi-\eta-1} \theta^{-\lambda/\xi} g_{\eta+\lambda}^{(r)} = k \sum_{\lambda=0}^{\lambda=\xi-\eta-1} k^{-\lambda/\zeta} g_{\eta+\lambda}^{(r)} \qquad \text{on } \gamma$$

We require the following conditions to be satisfied:

$$g_\eta^{(r)} = i\varphi, \qquad g_{\eta+\lambda}^{(r)} = 0 \qquad (\lambda = 1,2,\ldots,\ \xi-\eta-1) \qquad \text{on } \gamma \tag{6.6}$$

Then the next relations hold good

$$\theta g^{(r)} = ik\varphi \qquad \text{on } \gamma \tag{6.7}$$

(In the three last displayed equations r takes on the values $1, \ldots, (n - l)/2$).

We require that the principal parts of the change function of the fundamental integrals and the coefficients of expansion of these functions satisfy contour conditions analogous to those given in Section 5:

$$f_0^{(q)} = i\varphi, \quad \mathrm{Re}\,\{f_{0\alpha}^{(q)}\} > 0 \quad \text{on} \quad \gamma \tag{6.8}$$

$$(q = 1, \ldots \tfrac{1}{2} l)$$

$$f_{\varkappa+\lambda}^{(q)} = 0 \quad (\lambda = 0, 1, \ldots, \zeta - \varkappa - 1) \quad \text{on} \quad \gamma \tag{6.9}$$

The following contour relations will hold good:

$$kf^{(q)} = ik\varphi \quad \text{on} \quad \gamma \quad \left(q = 1, \ldots, \frac{1}{2} l\right) \tag{6.10}$$

6. The functions $g_{\eta+\lambda}$ are determined by equations of form (4.1) which have no singular points on γ (see Section 4). Functions $f_{\chi+\lambda}$ can be obtained from equations (3.1), which also have no singular points on γ, for f_0 can have no stationary points on γ owing to the boundary conditions. By hypothesis there are no singular points of L on γ, nor any points of common tangency of characteristics of the operator L belonging to distinct families. From this it follows that if the coefficients of (1.1) are sufficiently smooth, then the functions $g_{\eta+\lambda}$ and $f_{\chi+\lambda}$ will be sufficiently smooth in the neighborhood of γ, and hence the following conditions will hold good:

$$\mathrm{Re}\,\{g_\alpha^{(r)}\} > 0, \qquad \mathrm{Re}\,\{f_\alpha^{(q)}\} > 0 \quad \text{on} \quad \gamma$$

which guarantee the damping of all terms of the right-hand side (6.3) in the neighborhood of γ.

7. Substituting expression (6.3) into contour condition (5.1), and taking into account (6.7) and (6.10) by means of relation (1.7), we obtain the following equation

$$\sum_{q=1}^{q=l/2} k^\mu \sum_{v=0}^{v=\mu} k^{-v} \sum_{u=0}^{u=R}{}^{*} k^{-u} D_{v,q}^{(\mu,0)} (\Phi_u^{(q)}) +$$
$$+ \sum_{r=1}^{r=\frac{1}{2}(n-l)} k^\mu \sum_{v=0}^{v=\mu} k^{-v} \sum_{u=0}^{u=R}{}^{*} k^{-u} D_{v;r}^{(\mu,0)} (\Psi_u^{(r)}) = k^\mu g^{(\mu)} \text{ on } \gamma$$

On this basis contour conditions can be obtained for $\Phi_u (u < R)$ and Φ_R by the process briefly described in Section 5.

8. Thus, when $t < t_0$, problem A for equation (1.1) has solutions which decrease as we pass from γ into the interior of Γ. The rate of this decrease increases with an increase in k, i.e. with an increase in the speed of oscillation of the functions contained in the contour conditions.

When certain hypotheses analogous to propositions (a), (b) and (c), Section 5, are satisfied, then the boundedness of the remainder term* is guaranteed, and the solution of problem A can be constructed with an arbitrary degree of accuracy for large enough k. This construction can be carried out by the superposition of fundamental integrals and of integrals with supporting contours passing along boundary γ.

The approximate construction of fundamental integrals entering into the solution of problem A is reduced to the integration of equation (1.15) with contour conditions (6.8), equations (3.1) and (3.2) with contour conditions (6.9), and equation (3.3) for given contour values Φ_u. As in Section 5, all these operations can be reduced to the successive solutions of a certain number of Cauchy problems for first-order linear equations in the neighborhood of γ.

The approximate construction of the integrals with support contours, which enter into the solution of problem A, can be reduced to the integration of equations (1.17) with contour condition $f_0 = 0$, equations of type (4.1) with contour conditions (6.6), and equations (4.2) with given contour values $\Phi_u(u < R)$. The reduction of nonlinear equation (1.17) to a certain number of linear equations cannot be accomplished. However, since the integrals with the support contour γ, as well as the fundamental integrals, need only be constructed in a neighborhood of γ, and since for all practical purposes this neighborhood is quite narrow (see below), it follows that the method described in Section 4 for the solution of equation (1.17) is quite applicable.

9. The fundamental integrals which occur in the solution of problem A, when $t = 1/(n - l + \rho) < t_0$, can be constructed with asymptotic error of order $k^{-\rho+1}$ by means of approximating equation (3.4). The corrections which can be realized by the use of operator N will affect only the free terms of the equations and the boundary conditions determining the coefficients of the expansion of the change and intensity functions.

10. The damping of the fundamental integrals and of the integrals with support contour γ is of a nonuniform nature. The rate of damping of the fundamental integrals for $t < t_0$ is always less than the rate of damping of the integrals with the said support contour**, and moreover, the rate

* We note that hitherto it has not been necessary to require the operator N to be elliptic when $t < t_0$, but this will become necessary for proposition (c) to hold good.

** This fact served as the basis for introducing the term "fundamental integral" (an integral which penetrates deeper in the region).

of damping of the fundamental integral depends on the change index, while the rate of damping of the integrals with the said support contour remains stable.

When $t = 0$, the fundamental integrals lose the damping property. At the same time, the proposed method becomes inapplicable for constructing the fundamental integrals (since the parameter k ceases to be large), but the method is still valid for constructing the integrals with the said support contour (for θ still remains large).

When $t = 0$, the ordinary method of a small parameter can be used for constructing the fundamental integral. Problem A, for the case $t = 0$, was considered in great detail by Vishik and Liusternik in paper [3] (where it was assumed that the small parameter did not appear in the boundary conditions). For constructing the fundamental integrals, the method of successive approximations was used in paper [3], while for constructing the integrals with support contour γ (the boundary layer, in the terminology of [3]) a method was used similar to that explained here and applied earlier in monograph [2].

The Vishik and Liusternik results can, of course, be used in the case $t > 0$, but it should be realized that they become less effective with an increase of t. When $t \geqslant t_0$, the procedure used in [3] becomes entirely useless.

Remark: Vishik and Liusternik introduced the term "boundary layer" [pogransloi], interpreting it to mean any integral having the property of exponential damping. This property is possessed by integrals with a supporting contour. These two concepts are not identical, however. According to Vishik and Liusternik, all integrals forming the solution of problem A when $t < t_0$ must be attributed to the "boundary layer" [pogransloi], while according to the terminology adopted here, the solution of problem A consists of fundamental integrals and of integrals with a support contour γ. (The difference between these integrals is described above).

If $t < t_0$, the solution of problem A contains those integrals with the support contour γ which can be constructed only when the boundary of the region in question does not touch a characteristic of operator L. If γ does pass along the characteristics of L or even touch them, the properties of the solution of problem A will change radically. For example, this phenomenon explains the fact that in the theory of thin elastic shells the state of stress of an open cylindrical shell or of a shell with a hole is quite different from the state of stress of a closed cylindrical shell without a hole (the characteristics of operator L in this case coincide with the generators of the cylinder).

For a study of the problem in which the boundary of the region passes along the characteristics of operator L, a concept of the integral with a characteristic support contour was introduced in [2] (in [3] an analogous concept was called a parabolic limiting layer).

7. 1. Let us consider problem B for equation (1.1). By this we mean the integration of equation (1.1) with boundary conditions

$$D^{(\mu,0)}(\Phi) = g^{(\mu)}e^{ik\varphi} \quad \text{on } \gamma \quad (\mu = 0, 1, 2, \ldots, n-1)$$

Here γ is a curve which does not touch the characteristics of operator N and which coincides with the line $\alpha = \alpha_0$ (if necessary, by the use of a preliminary real transformation of the independent variables); $\phi(\beta)$ is a real function, $\phi'(\beta)$ is never zero. The parameters of the problem are assumed to be sufficiently smooth in the same sense as that given in Section 6. Problem B is solved in general by the same process as problem A. Moreover, the solution of problem B with a parameter has much in common with the solution of this problem without a parameter. We will shorten the explanations by taking the opportunity to make appropriate references to the relevant Sections of this paper and of the earlier work [1].

2. If the change index t of problem B is greater than t_0, then the solution of this problem can be constructed approximately by the superposition of fundamental integrals with the same change index, i.e. by means of the integrals corresponding to the families of characteristics of operator N. That is, we may set

$$\Phi = \sum_{q=1}^{q=n} \Phi_{*}^{(q)} e^{kf^{(q)}} \tag{7.1}$$

where the summation is carried out over all n families of the characteristics of N (N is assumed not to have multiple families of characteristics).

The following contour conditions have to be imposed on the change function and on the coefficients of the expansion of these functions:

$$f_0^{(q)} = i\varphi(\beta), \quad f_{\varkappa+\lambda} = 0 \quad \text{on } \gamma \quad (\lambda = 0, 1, 2, \ldots, \zeta - \varkappa - 1) \tag{7.2}$$

The contour conditions for the coefficients of expansion of the intensity function and for the remaining terms can then be derived in the same way as in the solution of problem B in the earlier article [1].

Determining the principal part of the change function and the coefficients of expansion of the change and intensity functions is reduced in the final step to the successive solution of Cauchy problems for first-order linear equations, just as in Section 5. These equations will have no singular points in the neighborhood of γ if we assume that there are no points of common tangency between the characteristics of N belonging to distinct families. The solution of problem B can be constructed to within an error of order $k^{-\rho+1}$ by means of the approximate equation (7.2).

If operator N is completely hyperbolic, then all the functions $f_0^{(q)}$ will be pure imaginary in consequence of (7.2). If, in addition, $t > 1/(n - l - 1)$, i.e. $\rho > 1$, then the functions $f_{\chi+\lambda}$ will be identically zero (see Section 2) and the solution of problem B will have the oscillating nature typical of such problems. However, if t is restricted to the interval

$$\frac{1}{n-l} = t_0 < t < \frac{1}{n-l-1}$$

then the functions $f_{\chi+\lambda}$ cannot be assumed to be zero in a neighborhood of γ lying in Γ, and the sum (7.1) will in general contain terms which will increase rapidly as $k \to \infty$.

3. If the change index of problem B is less than t_0, and if γ does not touch any characteristics of L or of N, then the approximate solution is composed of the sum of fundamental integrals corresponding to all the l families of characteristics of L, and of the sum of integrals with support contour γ corresponding to all the $n - l$ roots of equation (4.6).

The integral with support contour γ must be taken in the form (6.1), (6.2), while ξ and η are subjected to the relation (6.5). The coefficients of expansion of the change function of the integrals with support contour γ must be subjected to the conditions

$$g_\eta^{(r)} = i\varphi, \qquad g_{\eta+\lambda} = 0 \quad \text{на } \gamma \qquad (\lambda = 1, \ldots, \xi - \eta - 1 \quad r = 1, \ldots, {}^1/_2(n-l))$$

while the principal part of the change function of the fundamental integrals, and the coefficients of expansion of these functions, must be made to satisfy the following requirement:

$$f_0^{(q)} = i\varphi, \qquad f_{\varkappa+\lambda}^{(q)} = 0 \quad \text{на } \gamma\,(\lambda = 0, 1, \ldots, \zeta - \varkappa - 1,\ q = 1, \ldots, {}^1/_2 l))$$

In this case, the contour conditions which are put on the coefficients of expansion of the intensity function and on the remainder terms, as well as those on the fundamental integrals and on the integrals with support contour γ, are determined as in the case $t > t_0$. In the case in question, the construction of function (1.4) for the fundamental integrals reduces to the successive solution of a Cauchy problem for first-order equations, one of which (the one determining the principal part of the change function) is nonlinear. All these equations will have no singular points in the neighborhood of γ if the coefficients of equation (1.1) are sufficiently smooth, if this equation does not possess singular points on γ, and if on γ there are no points of common tangency of characteristics of L belonging to distinct families.

4. If $t < t_0$, then the solution of problem B will contain all integrals with support contour γ including those which in the neighborhood of γ increase rapidly as they are moved away from Γ. Therefore, for small t, the solution of problem B will have a purely oscillatory character only in exceptional cases, even when L and N are completely hyperbolic operators.

8. 1. Let us consider the problem of the construction of a particular integral of the equation

$$hN(\Phi) + L(\Phi) = \psi(\alpha, \beta)\, e^{kf(\alpha, \beta)} \tag{8.1}$$

where (as in the earlier article [1]) f is a pure imaginary sufficiently smooth function which has no stationary points in the region in question, ψ is a sufficiently smooth function (in general complex), while k is a sufficiently large constant.

Assuming that k and h are connected by the relation (1.2), we call t the change index of the free term.

2. Let us suppose that the change index of the free term is a rational number and satisfies one of the following three conditions:

$$t \leqslant \frac{1}{n-l+1}, \qquad t \geqslant \frac{1}{n-l-1}, \qquad t = \frac{1}{n-l} = t_0 \tag{8.2}$$

Then a particular integral of (8.1) can be sought in the form

$$\Phi = k^m \Phi_* e^{kf}, \qquad \Phi_* = \sum_{u=0}^{u=R}{}^{*} k^{-u}\Phi_u \tag{8.3}$$

where ζ is chosen so that $1/t$ will be a number of the form $\sigma + r/\zeta$. Substituting (8.3) into (8.1) we obtain a relation similar to (1.11):

$$k^{m+n-1/t}\Big\{\sum_{s=0}^{s=n+R}{}^{*}\sum_{p=0}^{p=s}{}^{*} k^{-s}N_{s-p}(\Phi_p)\Big\} + k^{m+l}\Big\{\sum_{r=0}^{r=l+R}{}^{*}\sum_{u=0}^{u=r}{}^{*} k^{-r}L_{r-u}(\Phi_u)\Big\} = \psi$$
$$(r-u \leqslant l; \quad u \leqslant R; \quad s-p \leqslant n; \quad p \leqslant R)$$

Following the earlier procedure, we equate the coefficients of equal powers of k on both sides of this equation. Three cases can then arise.

(a) If

$$l = n - 1/t + \rho \qquad (\rho \geqslant 1)$$

we set $m = -l$, and obtain

$$L_0\Phi_0 = \psi$$

$$L_0\Phi_r = -\sum_{u=0}^{u=r-1/\zeta}{}^{*} L_{r-u}(\Phi_u)$$

$(r - u < l;\ r = 0,\ 1/\zeta,\ 2/\zeta,\ \ldots,\ \theta - 1/\zeta;\ \theta$ is the smaller of the numbers ρ and $R)$

$$L_0\Phi_r = -\sum_{u=0}^{u=r-1/\zeta} L_{r-u}(\Phi_u) - \sum_{p=0}^{p=r-\rho}{}^{*} N_{r-p-\rho}(\Phi_p) \qquad (8.4)^1$$

$$(r-u\leqslant l;\quad r-p-\rho\leqslant n;\quad r=\rho,\ \rho+1/\zeta,\ \rho+2/\zeta,\ldots,R-1/\zeta)$$

$$\sum_{r=R}^{r=l+R}{}^{*}\ \sum_{u=0}^{u=R}{}^{*}\ k^{-r}L_{r-u}(\Phi_u) + \sum_{s=R+1}^{s=n+R+\rho}{}^{*}\ \sum_{p=0}^{p=s-\rho}{}^{*}\ k^{-s}N_{s-\rho-p}(\bar{\Phi}_p) = 0$$

$$(r-u\leqslant l;\quad s-p-\rho\leqslant n)$$

(b) If $l = n - 1/t - \rho\,(\rho > 1)$, we set $m = 1/t - n = -(l + \rho)$, and obtain the system

$$N_0\Phi_0 = \psi$$

$$N_0\Phi_s = -\sum_{p=0}^{p=s-1/\zeta}{}^{*} N_{s-p}(\Phi_p)$$

$(s - p \leqslant n,\ s = 0,\ 1/\zeta,\ 2/\zeta,\ \ldots,\ \theta - 1/\zeta;\ \theta$ is the smaller of the numbers ρ and $R)$

$$N_0\Phi_r = -\sum_{p=0}^{p=s-1/\zeta} N_{s-p}(\Phi_p) - \sum_{u=0}^{u=s-1/\zeta} N_{s-u-\rho}(\Phi_u) \qquad (8.5)^1$$

$$(s-p\leqslant n;\quad s-u-\rho\leqslant l;\quad s=\rho,\ \rho+1/\zeta,\ \rho+2/\zeta,\ldots,R-1/\zeta)$$

$$\sum_{s=R}^{s=n+R}{}^{*}\ \sum_{p=0}^{p=R}{}^{*}\ k^{-s}N_{s-p}(\Phi_p) + \sum_{r=R+1}^{r=l+R+\rho}{}^{*}\ \sum_{u=0}^{u=r-\rho} k^{-r}L_{r-\rho-u}(\Phi_u) = 0$$

$$(s-p\leqslant n;\quad r-\rho-u\leqslant l)$$

* These relations are valid only if $R > \rho$.

(c) If $l = n - 1/t$, then, setting $m = -l = 1/t - n$ and $\zeta = 1$, we obtain the system

$$(L_0 + N_0)\Phi_0 = \psi$$

$$(L_0 + N_0)\Phi_r = -\sum_{u=0}^{r=r-1/\zeta}{}^{*} L_{r-u}(\Phi_u) - \sum_{p=0}^{p=s-1/\zeta}{}^{*} N_{s-p}(\Phi_p) \tag{8.6}$$

$$(r - u \leqslant l;\quad s - p \leqslant n;\quad r, s = 1, 1 + 1/\zeta, \ldots, R - 1/\zeta)$$

$$\sum_{r=R}^{r=l+R}{}^{*} \sum_{u=0}^{u=R}{}^{*} k^{-r} L_{r-u}(\Phi_u) + \sum_{s=R}^{s=n+R}{}^{*} \sum_{p=0}^{p=R}{}^{*} k^{-s} N_{s-p}(\Phi_p) = 0$$

$$(r - u \leqslant l \quad s - p \leqslant n)$$

3. Systems (8.4), (8.5) and (8.6) generally make it possible to determine successively all expansion coefficients of the intensity function by means of algebraic operations. But to be able to do this it must be required that the following conditions hold good at all points of the region in question:

(a) If $t < 1/(n - l + 1)$, the expression L_0 has to be different from zero, m.e. the level lines of the change function of the free term must not touch the characteristics of operator L;

(b) if $t > 1/(n - l - 1)$, the expression N_0 has to be different from zero, i.e. the level lines of the change function of the free term must not touch the characteristics of operator N;

(c) if $t = 1/(n - l)$, the expression $L_0 + N_0$ must not be zero (this requirement does not have a simple geometrical interpretation).

4. If the expressions L_0, N_0, and $L_0 + N_0$ are identically zero for the corresponding values of t and in the region in question, then we have a case analogous to the resonance case, i.e. the value of index m must be increased by unity (if the characteristics of the corresponding operator are simple, not multiple).

The expansion coefficients of the intensity function are then determined by linear first-order differential equations. We will not dwell upon the details here. They are to be found in the earlier article [1].

9. The results presented admit of various generalizations.

1. The generalization to the case of more than two independent variables is trivial. The few assertions which require any examination are those based on the expansions of the left-hand sides of equations (1.15)

and (1.16) into factors which are linear with respect to $f_{0\alpha}$ and $f_{0\beta}$.

2. The generalization to the case when operators L (for $t < t_0$) and N(for $t > t_0$) have multiple families of characteristics can be obtained without changing the form (1.3) of the final integral. Here, however, the selection of fraction χ/ζ requires careful analysis involving the consideration of a large number of various possibilities.

3. The generalization to the case when in place of equation (1.1) we have a system of linear differential equations presents no fundamental difficulties. Here the method of the selection of noncontradictory values of the index of intensity described in monograph [2] can be used.

Remark. An example, constructed by Hadamard, on the instability of the solution of Cauchy's problem for an elliptic equation is widely known (for example, see [4]). It can obviously be obtained as a particular case of the solution considered in paper [1] of problem B for the elliptic equation without a parameter, when the initial conditions contain a rapidly oscillating function. Lax [5] has pointed out that this phenomenon reveals itself quite naturally in the application of asymptotic integration: it merely relates to the fact that the change function proves pure imaginary only for an entirely hyperbolic equation. The results obtained here in Section 7, subsections 2, 4, show that in solving Cauchy's problem for an equation of the hyperbolic type with a small principal part, phenomena can also occur, which approach the phenomenon revealed in Hadamard's example. This means that if the contour condition formulated for problem B in Section 7, is fixed, a value of h can be selected in equation (1.1) so small that Φ becomes larger than any preassigned number at points near γ and inside Γ.

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